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Two and three electrons in a quantum dot: $1/|J|$-expansion

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Abstract. We consider systems of two and three electrons in a two-dimensional parabolic quantum dot. A magnetic field is applied perpendicularly to the plane of motion of the electron. We show that the energy levels corresponding to states with high angular momentum, J, and a low number of vibrational quanta may be systematically computed as power series in $1/|J|$. These states are relevant in the high-B limit.

1. Introduction

Recently, few-electron systems in nearly two-dimensional semiconductor structures has been a subject of intense theoretical and experimental research [1]. In the present paper, we continue a programme, initiated in [2, 3], aimed at providing reliable analytical estimates for the energy levels of model two-dimensional systems on the basis of a simple physical picture for the states.

The idea is to use the inverse of the angular momentum as an expansion parameter in the Schrödinger equation, thus obtaining nonperturbative series for the energy and the wave function. Variants of this method have been widely used in atomic and nuclear physics [4, 5].

The static exciton in a magnetic field was considered in [2]. Energy levels were computed as a function of the mass ratio and the magnetic field. Comparison with estimates obtained from two-point Padé approximants [6] yielded excellent results at any magnetic field strength.

In reference [3], a model of three electrons in a quantum dot with $1/r^2$ repulsion, considered previously in [7], was studied. The $1/|J|$-method provided analytical estimates for the energy and a set of approximate quantum numbers to label the states. We will make use below of many of the results obtained in [3].

Systems of two and three electrons in a two-dimensional parabolic quantum dot are to be studied in the present paper. A magnetic field is applied perpendicularly to the dot's plane. Numerical calculations for these systems have been reported elsewhere [8–10]. In this problem, there is only one relevant parameter, $\beta = (E_{coul}/\hbar\Omega)^{1/6}$, where E_{coul} is the characteristic Coulomb energy, and Ω is expressed in terms of the dot and the cyclotron frequencies as $\Omega = \sqrt{(\omega_0^2 + \omega_c^2/4)}$. $\beta \to 0$ will be called the oscillator limit, and $\beta \to \infty$ will be called the Wigner limit, suggesting that we are dealing with a few-electron version of the Wigner solid. In typical quantum dot experiments, $\beta \sim 1$.

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The physical picture emerging from the $1/|J|$-expansion is the following. In the leading approximation, the electrons are located at the vertices of a regular polygon, whose side minimizes the total classical energy (centrifugal plus potential). The first corrections take account of small (harmonic) vibrations around the equilibrium configuration. Higher corrections come from anharmonic oscillations. This picture is supported by numerical calculations for few-electron systems not only in the strong-coupling regime [11], but also in the region where $\beta \sim 1$ [12].

Reference [12] deserves special comment; in [12] the few-electron problem in Eckardt coordinates is treated quasiclassically. Although the starting points are far apart, our results are very similar up to the harmonic approximation. The main difference between reference [12] and our work is that the expansion parameter is $\overline{h}$ in [12], and is $1/|J|$ in our work. Higher orders of the quasiclassical expansion are not reported in [12].

The plan of the paper is as follows. In section 2, we present the $1/|J|$-method. Results for two and three electrons are shown in sections 3 and 4. Concluding remarks are given in the last section.

2. The $1/|J|$-expansion

We start with the dimensionless Hamiltonian governing the two-dimensional motion of N electrons in a quantum dot of energy $\overline{h}\omega_0$, and in a magnetic field normal to the plane of motion:

$$\frac{H}{\overline{h}\Omega} = \frac{1}{2}\sum_{k=1}^{N}(p_k^2 + r_k^2) + \beta^3 \sum_{k<l}\frac{1}{r_{kl}} + \frac{\omega_c}{2\Omega}\sum_{k=1}^{N} J_k + \frac{g\omega_c}{2\Omega}S_z. \tag{1}$$

The conventions are as follows. μ is the electron effective mass, $\omega_c = eB/\mu c$ is the cyclotron frequency, g is the effective gyromagnetic factor, S_z is the z-component of the total spin of the system, and $\Omega = \sqrt{(\omega_0^2 + \omega_c^2/4)}$ is the effective dot frequency. The length unit is $\sqrt{\overline{h}/(\mu\Omega)}$. The parameter β is given by

$$\beta^3 = \sqrt{\frac{\mu e^4}{\kappa^2\overline{h}^2}\Big/\overline{h}\Omega}$$

where κ denotes the dielectric constant.

The actual parameter governing the problem is β. When $\beta \ll 1$, the levels are small perturbations around oscillator energies. On the other hand, when $\beta \to \infty$, the energy minimum is reached in a classical configuration which can be seen as a few-body model of a Wigner crystal. Both limiting situations reduce to exactly solvable problems.

Through the introduction of Jacobi coordinates, the centre-of-mass motion is separated from the internal motion:

$$\frac{H}{\overline{h}\Omega} = \frac{H_{cm}}{\overline{h}\Omega} + \frac{H_{int}}{\overline{h}\Omega} \tag{2}$$

where

$$\frac{H_{cm}}{\overline{h}\Omega} = -\left(\frac{\partial^2}{\partial\rho_{cm}^2} + \frac{1}{\rho_{cm}}\frac{\partial}{\partial\rho_{cm}}\right) + \frac{J_{cm}^2}{\rho_{cm}^2} + \frac{1}{4}\rho_{cm}^2 + \frac{\omega_c}{2\Omega}J_{cm} \tag{3}$$

$$\frac{H_{int}}{\overline{h}\Omega} = -\sum_{k=1}^{N-1}\left(\frac{\partial^2}{\partial\rho_k^2} + \frac{1}{\rho_k}\frac{\partial}{\partial\rho_k}\right) - \sum_{k=1}^{N-2}\left(\frac{1}{\rho_k^2} + \frac{1}{\rho_{k+1}^2}\right)\frac{\partial^2}{\partial\theta_k^2}$$
$$+ 2\sum_{k=1}^{N-3}\frac{1}{\rho_{k+1}^2}\frac{\partial^2}{\partial\theta_k\,\partial\theta_{k+1}} + \frac{2\mathrm{i}J}{N-1}\sum_{k=1}^{N-2}\left(\frac{1}{\rho_k^2} - \frac{1}{\rho_{k+1}^2}\right)\frac{\partial}{\partial\theta_k}$$

$$+\sum_{k=1}^{N-1}\left(\frac{J^2}{(N-1)^2}\frac{1}{\rho_k^2}+\frac{1}{4}\rho_k^2\right)+\beta^3\sum_{k<l}\frac{1}{r_{kl}}+\frac{\omega_c}{2\Omega}J+g\frac{\omega_c}{2\Omega}S_z. \tag{4}$$

We have written H_{int} in terms of rotationally invariant coordinates, i.e. the moduli of the Jacobi vectors and the angles between them. The Jacobi vectors were defined as

$$\boldsymbol{\rho}_k=\sqrt{\mu_k/\mu_1}\left\{\boldsymbol{r}_{k+1}-\frac{1}{k}\sum_{l=1}^{k}\boldsymbol{r}_l\right\}\qquad k=1,2,\ldots,N-1 \tag{5}$$

where the reduced masses are $\mu_k=k/(k+1)$. The angle between $\boldsymbol{\rho}_k$ and $\boldsymbol{\rho}_{k+1}$ is denoted as θ_k. J labels the total internal angular momentum.

The eigenvalues of H_{cm} are simply

$$\frac{E_{cm}}{\hbar\Omega}=2n_{cm}+|J_{cm}|+1+\frac{\omega_c}{2\Omega}J_{cm}. \tag{6}$$

We shall obtain approximate expressions for the eigenvalues of H_{int}. We consider states with high $|J|$-values, which are the relevant states at high magnetic fields. It is intuitively evident that at high $|J|$, $E_{int}\sim|J|$ and $\langle\rho_k^2\rangle\sim|J|$. The solution of the Schrödinger equation may be organized as a power series in $1/|J|$. A scaling of dimensions such that $\rho^2\to|J|\rho^2$ makes evident the dependence of each term on $|J|$. The scaled Hamiltonian is written as

$$\begin{aligned}h=\frac{1}{|J|}&\left\{\frac{H_{int}}{\hbar\Omega}-\frac{\omega_c}{2\Omega}J-\frac{g\omega_c}{2\Omega}S_z\right\}\\&=\sum_{k=1}^{N-1}\left\{\frac{1}{(N-1)^2}\frac{1}{\rho_k^2}+\frac{1}{4}\rho_k^2\right\}+\tilde{\beta}^3\sum_{k<l}\frac{1}{r_{kl}}\\&\quad+\frac{1}{J^2}\left\{-\sum_{k=1}^{N-1}\left(\frac{\partial^2}{\partial\rho_k^2}+\frac{1}{\rho_k}\frac{\partial}{\partial\rho_k}\right)-\sum_{k=1}^{N-2}\left(\frac{1}{\rho_k^2}+\frac{1}{\rho_{k+1}^2}\right)\frac{\partial^2}{\partial\theta_k^2}\right.\\&\quad\left.+\;2\sum_{k=1}^{N-3}\frac{1}{\rho_{k+1}^2}\frac{\partial^2}{\partial\theta_k\,\partial\theta_{k+1}}\right\}+\frac{2\mathrm{i}}{J(N-1)}\sum_{k=1}^{N-2}\left(\frac{1}{\rho_k^2}-\frac{1}{\rho_{k+1}^2}\right)\frac{\partial}{\partial\theta_k}\end{aligned} \tag{7}$$

where we have introduced the 'renormalized' coupling constant $\tilde{\beta}^3=\beta^3/|J|^{3/2}$. When taking the formal limit $|J|\to\infty$, $\tilde{\beta}^3$ is kept fixed to take account of Coulomb repulsion nonperturbatively.

In what follows, we consider only the two- and three-electron problems. The only term surviving in the r.h.s. of (7) when $|J|\to\infty$ is the effective potential. Its absolute minimum gives a classical contribution to the energy. It is reached in a configuration where the particles sit at the corners of a regular polygon. In this configuration, the effective potential is a function of ρ_1 only:

$$U_{eff}=\frac{2\sin^2\pi/N}{N\rho_1^2}+\frac{N\rho_1^2}{8\sin^2\pi/N}+\frac{\tilde{\beta}^3\sin\pi/N}{\rho_1}\sum_{i<j}\frac{1}{|\sin\theta_{ij}/2|} \tag{8}$$

where θ_{ij} is the angle between particles i and j, measured from the c.m. Minimization of U_{eff} leads to an equilibrium value for ρ_1, ρ_{10}. The equilibrium values of the other coordinates, ρ_{k0}, θ_{k0}, are determined from geometry.

Higher contributions to the energy come from relaxing the equilibrium configuration, $\rho_k=\rho_{k0}+y_k/|J|^{1/2}$, $\theta_k=\theta_{k0}+z_k/|J|^{1/2}$. The r.h.s. of (7) may, thus, be expanded as

$$h=h_0+\frac{h_2}{|J|}+\frac{h_3}{|J|^{3/2}}+\frac{h_4}{|J|^2}+\cdots \tag{9}$$

where $h_0 = U_{eff}(\rho_{10})$, h_2 describes in general harmonic oscillations in effective magnetic fields, and h_3, h_4, etc account for anharmonicities.

For the spatial wave function and the scaled energy (related to h), we write series like equation (9):

$$\psi = \psi_0 + \frac{\psi_1}{|J|^{1/2}} + \frac{\psi_2}{|J|} + \cdots \tag{10}$$

$$\epsilon = \epsilon_0 + \frac{\epsilon_2}{|J|} + \frac{\epsilon_4}{J^2} + \cdots. \tag{11}$$

These expressions are substituted into the Schrödinger equation, leading to the chain of uncoupled equations

$$h_0 = \epsilon_0 \tag{12}$$

$$h_2\psi_0 = \epsilon_2\psi_0 \qquad \text{etc.} \tag{13}$$

Higher corrections to the energy are obtained from ordinary perturbation theory, where h_3, h_4, etc are interpreted as perturbations.

Below, we present results for two and three electrons.

3. Two electrons

For two particles, there is only one Jacobi coordinate, $\boldsymbol{\rho}_1 = \boldsymbol{r}_2 - \boldsymbol{r}_1$. Its equilibrium value (the modulus) satisfies the equation

$$0 = -2 + \frac{1}{2}\rho_{10}^4 - \tilde{\beta}^3\rho_{10}. \tag{14}$$

Writing $\rho_1 = \rho_{10} + y_1/|J|^{1/2}$, and substituting into the r.h.s. of (7), we obtain the operator coefficients h_2, h_3, etc. In particular

$$h_2 = -\frac{\mathrm{d}^2}{\mathrm{d}y_1^2} + \frac{1}{2}U''(\rho_{10})y_1^2 = \omega_1\left(a_1^\dagger a_1 + \frac{1}{2}\right) \tag{15}$$

where $\omega_1 = \sqrt{2U''(\rho_{10})} = \sqrt{3 + 4/\rho_{10}^4}$, $a_1 = \omega^{1/2}y_1/2 + \mathrm{i}p_1/\omega^{1/2}$, and $a_1^\dagger = \omega^{1/2}y_1/2 - \mathrm{i}p_1/\omega^{1/2}$. Note that in the oscillator limit $\omega_1 \to 2$, whereas in the Wigner limit ω_1 approaches the classical result $\sqrt{3}$ [13].

To this order, the spatial wave function is written as $\mathrm{e}^{\mathrm{i}J\theta}\psi_0(y_1)$, where $\psi_0 \sim (a_1^\dagger)^n|0\rangle$, and θ is the polar angle associated with the vector $\boldsymbol{\rho}_1$. Under a permutation of particles, θ changes by π, and thus even $|J|$ correspond to unpolarized spin states, $S = 0$, and odd $|J|$ correspond to polarized states, $S = 1$.

The operators h_k, $k \geqslant 3$, can be written in general as

$$h_k = \frac{(-1)^k}{\rho_{10}^{k-2}}\left\{\left(\frac{k-1}{\rho_{10}^4} + \frac{1}{2}\right)y_1^k + y_1^{k-3}\frac{\partial}{\partial y_1}\right\}. \tag{16}$$

Their matrix elements can be straightforwardly computed. Let us write out the results for the first nonzero terms of the series (11) in the present case:

$$\epsilon_0 = \frac{3}{4}\rho_{10}^2 - \frac{1}{\rho_{10}^2} \tag{17}$$

$$\epsilon_2 = \omega_1\left(n + \frac{1}{2}\right) \tag{18}$$

$$\epsilon_4 = \frac{3}{\rho_{10}^2\omega_1^2}\left(\frac{3}{\rho_{10}^4}+\frac{1}{2}\right)(2n^2+2n+1) - \frac{1}{4\rho_{10}^2} - \frac{1}{\rho_{10}^2\omega_1^4}\left(\frac{2}{\rho_{10}^4}+\frac{1}{2}\right)^2(30n^2+30n+11) \tag{19}$$

$$\begin{aligned}\epsilon_6 = {} & -\frac{15}{8\rho_{10}^4\omega_1^9}(4/\rho_{10}^4+1)^4(2n+1)(47n^2+47n+31) \\ & + \frac{9}{2\rho_{10}^4\omega_1^7}(4/\rho_{10}^4+1)^2(6/\rho_{10}^4+1)(2n+1)(25n^2+25n+19) \\ & - \frac{1}{2\rho_{10}^4\omega_1^5}(6/\rho_{10}^4+1)^2(2n+1)(87n^2+87n+86) \\ & + \frac{10}{\rho_{10}^{12}\omega_1^5}(2n+1)(14n^2+14n+13) \\ & + \frac{1}{\rho_{10}^4\omega_1^3}(2n+1)\{(10/\rho_{10}^4+1)(5n^2+5n+9)-9/\rho_{10}^4\} \\ & - \frac{3}{4\rho_{10}^4\omega_1}(2n+1) \qquad \text{etc.}\end{aligned} \tag{20}$$

n labels the number of excitation quanta. It is a good approximate quantum number of the problem over the whole interval $0 \leqslant \beta < \infty$. Notice that all of the ϵ_k with $k > 2$ go to zero in both the oscillator and Wigner limits. Thus, the $1/|J|$-expansion provides a nice interpolation scheme.

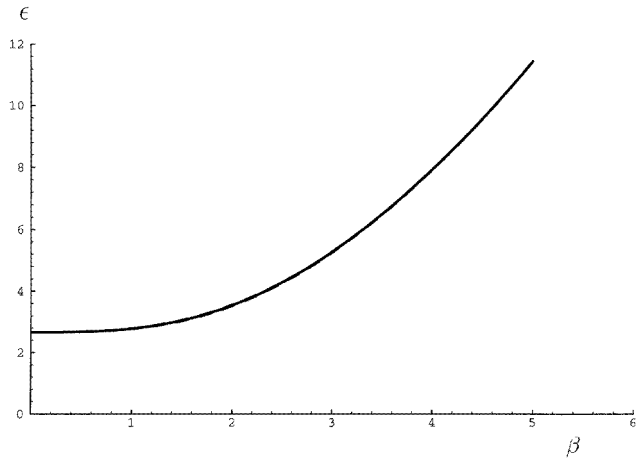
In figure 1(a), we compare different approximations to the r.h.s. of (7) computed according to the series (11). The excited state with $n = 2$ and $|J| = 3$ is shown. The convergence is excellent. We cannot distinguish the curves containing the corrections ϵ_4 and ϵ_6 (printed as dashed lines) from the solid line corresponding to $\epsilon_0 + \epsilon_2/|J|$. Even at such a low value of $|J|$ as 3, ϵ_4 introduces corrections lower than 1%, and ϵ_6 introdudes corrections lower than 0.3%. The relative weight of ϵ_6 in $\epsilon = \epsilon_0+\epsilon_2/|J|+\epsilon_4/J^2+\epsilon_6/|J|^3$ is shown in figure 1(b).

In figure 2, the energy $|J|\epsilon$ is compared with the estimate obtained in [14] from the two-point Padé approximant $P_{6,5}(\beta)$. These approximants are exact in both the $\beta \to 0$ and $\beta \to \infty$ limits. They were checked against exact results at particular intermediate values of β [15]. We expect the approximants to approach the exact energy from above, and to differ from it by no more than a few parts in 10^3 [14]. The results are consistent with the expectations. Similar results are revealed in the comparison with the numerical calculations of [9].

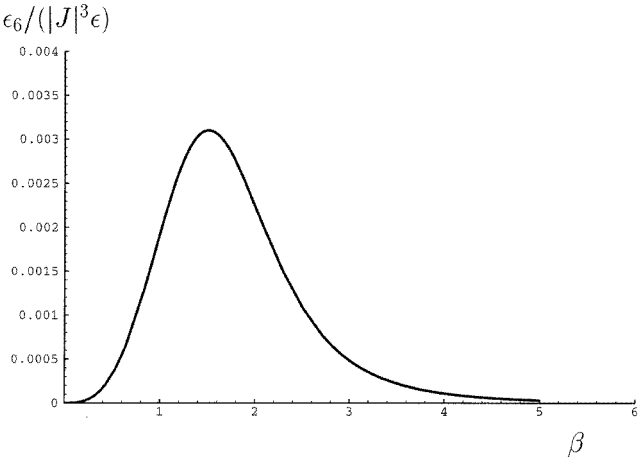
Notice that the conditions for the application of perturbation theory are fulfilled at any β when $|J| \geqslant 3$, i.e. the distances between adjacent levels, computed from $\epsilon_0+\epsilon_2/|J|$, are much greater than the corrections $\epsilon_4/|J|^2+\cdots$. Below, we will see that in the three-body problem the corrections, although small, may cause a partial rearrangement of the spectrum.

4. Three electrons

In the three-electron system, there are two Jacobi vectors, $\boldsymbol{\rho}_1 = \boldsymbol{r}_2 - \boldsymbol{r}_1$ and $\boldsymbol{\rho}_2 = \sqrt{4/3}\{\boldsymbol{r}_3 - (\boldsymbol{r}_1+\boldsymbol{r}_2)/2\}$, in terms of which we define two distances, ρ_1, ρ_2, and one angle, θ_1. The equilibrium configuration is an equilateral triangle ($\rho_{10} = \rho_{20}$, $\theta_{10} = \pm\pi/2$),



(a)



(b)

Figure 1. (a) The energy of the $|J| = 3$, $n = 2$ state of two electrons. (b) The relative weight of ϵ_6 in ϵ.

whose side satisfies the equation

$$0 = -1 + \rho_{10}^4 - 3\tilde{\beta}^3\rho_{10}. \tag{21}$$

At its minimum, $U_{eff}(\rho_{10}) = 3\rho_{10}^2/2 - \rho_{10}^{-2}/2$, thus providing a leading approximation to the energy:

$$\epsilon_0 = \frac{3}{2}\rho_{10}^2 - \frac{1}{2\rho_{10}^2}. \tag{22}$$

Expanding around one of the two equivalent configurations, i.e. fixing $\theta_{10} = \pi/2$ for example, we obtain a series like (9) in which the first operator coefficients are given by

$$h_2 = -\left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} + \frac{2}{\rho_{10}^2}\frac{\partial^2}{\partial z_1^2}\right) - \frac{2\mathrm{i}}{\rho_{10}^3}\,\mathrm{sgn}(J)(y_1 - y_2)\frac{\partial}{\partial z_1} + \frac{1}{4}\left(\frac{3}{\rho_{10}^4} + 1\right)(y_1^2 + y_2^2)$$

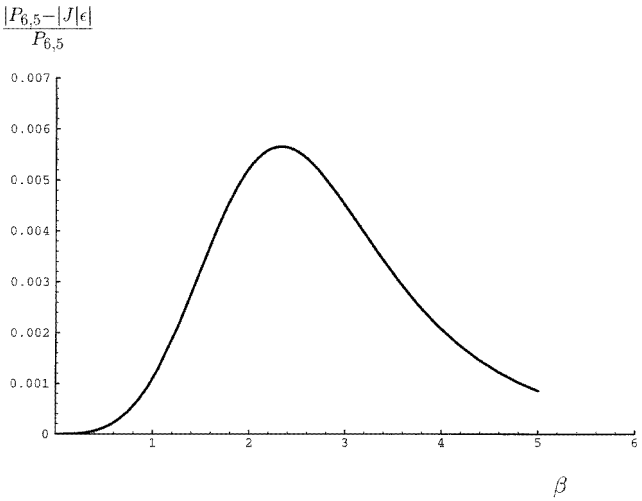


Figure 2. The relative difference between $|J|\epsilon$ and the $P_{6,5}$ Padé approximant for two electrons in a state with $|J| = 3$, $n = 0$.

$$+\frac{1}{16}\left(1-\frac{1}{\rho_{10}^4}\right)(5y_1^2+6y_1y_2+5y_2^2+3\rho_{10}^2z_1^2) \tag{23}$$

$$h_3=-\frac{1}{\rho_{10}}\left(\frac{\partial}{\partial y_1}+\frac{\partial}{\partial y_2}\right)+\frac{2}{\rho_{10}^3}(y_1+y_2)\frac{\partial^2}{\partial z_1^2}+\frac{3\mathrm{i}}{\rho_{10}^4}\,\mathrm{sgn}(J)(y_1^2-y_2^2)\frac{\partial}{\partial z_1}-\frac{1}{\rho_{10}^5}(y_1^3+y_2^3)$$
$$-\frac{1}{64\rho_{10}}\left(1-\frac{1}{\rho_{10}^4}\right)(19y_1^3+3y_1^2y_2+33y_1y_2^2+9y_2^3$$
$$-9\rho_{10}^2y_1z_1^2+21\rho_{10}^2y_2z_1^2) \tag{24}$$

$$h_4=\frac{1}{\rho_{10}^2}\left(y_1\frac{\partial}{\partial y_1}+y_2\frac{\partial}{\partial y_2}\right)-\frac{3}{\rho_{10}^4}(y_1^2+y_2^2)\frac{\partial^2}{\partial z_1^2}$$
$$-\frac{4\mathrm{i}}{\rho_{10}^5}\,\mathrm{sgn}(J)(y_1^3-y_2^3)\frac{\partial}{\partial z_1}+\frac{5}{4\rho_{10}^6}(y_1^4+y_2^4)$$
$$+\frac{1}{256\rho_{10}^2}\left(1-\frac{1}{\rho_{10}^4}\right)\left(\frac{329}{4}y_1^4-25y_1^3y_2+\frac{123}{2}y_1^2y_2^2+135y_1y_2^3+\frac{9}{4}y_2^4\right.$$
$$\left.-\frac{99}{2}\rho_{10}^2y_1^2z_1^2+27\rho_{10}^2y_1y_2z_1^2+\frac{141}{2}\rho_{10}^2y_2^2z_1^2+\frac{41}{4}\rho_{10}^4z_1^4\right)\qquad\text{etc.} \tag{25}$$

The Hamiltonian h_2, which is to be taken as the zeroth-order Hamiltonian, describes a harmonic oscillator of frequency $\omega_1=\sqrt{(3+1/\rho_{10}^4)}$ in the variable $y_s=(y_1+y_2)/\sqrt{2}$, plus a combination of harmonic oscillators and an effective magnetic field in the variables $y_m=(y_1-y_2)/\sqrt{2}$ and $z_m=\rho_{10}z_1/\sqrt{2}$. Note that the 'magnetic field' comes from the coupling between the angular momentum and the derivative with respect to the variable θ_1 (see the Hamiltonian (4) for $N=3$). We may use a symmetric gauge to describe the effective magnetic field. The Hamiltonian is transformed according to $h'=\mathrm{e}^{\mathrm{i}f}h\mathrm{e}^{-\mathrm{i}f}$, where $f=\mathrm{sign}(J)y_mz_m/(2\rho_{10}^2)$. h_2' takes the symmetric form

$$h_2'=-\frac{\partial^2}{\partial y_s^2}+\frac{1}{4}\left(3+\frac{1}{\rho_{10}^4}\right)y_s^2-\left(\frac{\partial^2}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial}{\partial\xi}+\frac{1}{\xi^2}\frac{\partial^2}{\partial\alpha^2}\right)$$

$$-\mathrm{i}\,\frac{\mathrm{sgn}(J)}{\rho_{10}^2}\frac{\partial}{\partial\alpha}+\frac{1}{8}\left(3-\frac{1}{\rho_{10}^4}\right)\xi^2 \tag{26}$$

where $\xi = \sqrt{y_m^2+z_m^2}$ and $\alpha = \arctan(z_m/y_m)$. The corresponding eigenvalues and eigenfunctions are

$$\epsilon_2 = \omega_1(n_s+1/2)+\omega_2(2n+|m|+1)+\mathrm{sgn}(J)m\omega_3 \tag{27}$$

$$\psi_0 \sim H_{n_s}(\sqrt{\omega_1/2}\,y_s)\mathrm{e}^{-\omega_1 y_s^2/4}\xi^{|m|}\mathrm{e}^{-\omega_2\xi^2/4}L_n^{|m|}(\omega_2\xi^2/2)\mathrm{e}^{\mathrm{i}m\alpha} \tag{28}$$

in which $\omega_2 = \sqrt{3/2-\rho_{10}^{-4}/2}$, $\omega_3 = 1/\rho_{10}^2$, H_{n_s}, and $L_n^{|m|}$ are Hermite and generalized Laguerre polynomials, and n and m are respectively principal and magnetic quantum numbers in the effective magnetic field. In the oscillator limit, $\omega_1 \to 2$, ω_2, $\omega_3 \to 1$, whereas in the Wigner limit, $\rho_{10} \to \infty$, $\omega_1 \to \sqrt{3}$, $\omega_2 \to \sqrt{3/2}$, and $\omega_3 \to 0$. At finite $\tilde{\beta}$ there is no degeneracy between the states of h_2; thus we may use nondegenerate perturbation theory to compute corrections.

We shall identify which of the levels (J, n_s, n, m) may describe electron states. This set of quantum numbers will serve as approximate quantum numbers for the three-electron problem. At intermediate couplings, a small mixing between the states of h_2' will be induced by the corrections h_3' and h_4'.

Let us recall the symmetry requirements for three-electron wave functions. A spatially antisymmetric wave function, $\Psi_0 = \mathrm{e}^{\mathrm{i}J\Xi}\psi_0$, where Ξ accounts for global rotations, corresponds to a spin-polarized state, $S = 3/2$, whereas a mixed-symmetry Ψ_0 is related to an unpolarized spin state, $S = 1/2$. Symmetry transformations will be expressed in terms of the variables that we are using, i.e. y_s, ξ, and α. Additionally, to identify physical states, Ψ_0 will be compared with oscillator functions for $\tilde{\beta} \to 0$. A detailed analysis is presented in the appendix of paper [3], to which we refer the reader. The main conclusion is that in the (n, m) plane, the spatially symmetric and antisymmetric states occupy the lines $m = J+3k$, where k is an integer. For example, let us consider the first states with $J = -3$ when $\tilde{\beta} \ll 1$. The lowest state is $(n_s, n, m) = (0, 0, 0)$. This corresponds to both one antisymmetric and one symmetric state. In our scheme, as tunnelling effects are not included, they are degenerate in energy. Of course, only the antisymmetric state may correspond to a state of three electrons. Very near to the lowest state there is the mixed-symmetry doublet, $(0, 0, 1)$. At excitation energies near 2, there are the levels $(1, 0, 0)$, $(1, 0, 1)$, $(0, 0, -1)$, $(0, 1, 0)$, and $(0, 1, 1)$. Symmetric and antisymmetric states have $m = 0$. The next set of states is at excitation energies near 4, etc.

Up to this order, the energy computed from $\epsilon = \epsilon_0+\epsilon_2/|J|$ leads to results very similar to that reported in [12]. That is, when $\beta \sim 1$ the relative error of the lowest state with a given $|J|$ represents only a few per cent of the total energy at $|J| = 3$, and decreases considerably as $|J|$ is increased. The reader may look at figures 3 and 4 of paper [12] as examples.

Let us study ϵ when β is varied from 0 to ∞. As in the two-electron problem, the higher corrections, $\epsilon_4/|J|^2+\cdots$, go to zero in both the $\beta \to 0$ and $\beta \to \infty$ limits. However, for certain levels, a rearrangement of the spectrum may take place. For example, let us consider spatially antisymmetric states with $J = -3k$, where k is positive and high enough. The lowest state in this sector has quantum numbers $(0, 0, 0)$. We show in figure 3 the excitation energies of three states with quantum numbers $(0, 0, 3)$, $(1, 0, 0)$, and $(0, 1, 0)$. Their energies are $3(\omega_2-\omega_3)$, ω_1, and $2\omega_2$ respectively. Transitions among these states induced by h_3', h_4', etc may occur at $\tilde{\beta} \geqslant 0.6$, thus resulting in a rearrangement of levels. To compute their energies at the intermediate values of β where the intersections occur, we shall resolve the degeneracy by means of degenerate perturbation theory. On the other

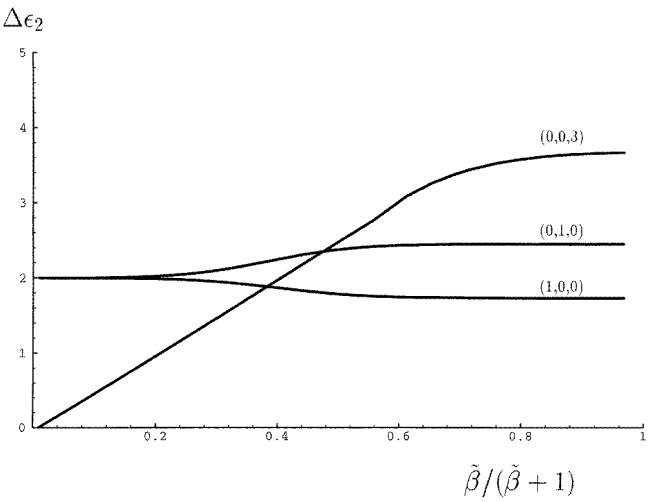


Figure 3. Excitation energies of three states with $J = -3k$, and k positive and high enough. The quantum numbers are indicated.

hand, there are levels, in particular those with numbers $(0, 0, 0)$, for which we can apply the $1/|J|$-expansion continuously from $\tilde{\beta} = 0$ to ∞.

Let us compute the coefficient ϵ_4 for a state with quantum numbers $(0, 0, 0)$. We assume $J < 0$ and $|J|$ large enough. These are the relevant states at high magnetic fields. We shall compute the matrix elements entering the expression

$$\epsilon_4 = \langle 0,0,0|h_4'|0,0,0\rangle - \sum_{n_s,n,m} \frac{\langle 0,0,0|h_3'|n_s,n,m\rangle\langle n_s,n,m|h_3'|0,0,0\rangle}{n_s\omega_1 + (2n+|m|)\omega_2 - m\omega_3}. \tag{29}$$

Taking into account the explicit form of h_3', i.e.

$$h_3' = A\frac{\partial}{\partial y_s} + By_s + Cy_s^3 + D \tag{30}$$

where

$$A = -\frac{\sqrt{2}}{\rho_{10}} \tag{31}$$

$$B = \frac{\sqrt{2}}{\rho_{10}}\left\{\frac{\partial^2}{\partial z_m^2} + 2\mathrm{i}\,\frac{\mathrm{sgn}(J)}{\rho_{10}^2}y_m\frac{\partial}{\partial z_m} - \frac{1}{4\rho_{10}^4}y_m^2 - \frac{3}{16}\left(1-\frac{1}{\rho_{10}^4}\right)(y_m^2+z_m^2)\right\} \tag{32}$$

$$C = -\frac{\sqrt{2}}{4\rho_{10}}\left(1+\frac{1}{\rho_{10}^4}\right) \tag{33}$$

$$D = -\frac{5\sqrt{2}}{32\rho_{10}}\left(1-\frac{1}{\rho_{10}^4}\right)(y_m^3 - 3y_m z_m^2) \tag{34}$$

we obtain that the sum over intermediate states in (29) contains only a few terms. The following result for ϵ_4 arises:

$$\epsilon_4 = \langle 0,0,0|h_4'|0,0,0\rangle - \frac{C^2}{3\omega_1}\langle 3|y_s^3|0\rangle^2$$

$$
\begin{aligned}
&- \frac{1}{3}\langle 0,3|D|0,0\rangle^2\left(\frac{1}{\omega_2-\omega_3}+\frac{1}{\omega_2+\omega_3}\right)+\frac{A^2}{\omega_1}\langle 1|\frac{\partial}{\partial y_s}|0\rangle^2\\
&- \frac{2C}{\omega_1}\langle 1|y_s|0\rangle\langle 1|y_s^3|0\rangle\langle 0,0|B|0,0\rangle\\
&- \frac{C^2}{\omega_1}\langle 1|y_s^3|0\rangle^2-\frac{1}{\omega_1}\langle 1|y_s|0\rangle^2\langle 0,0|B|0,0\rangle^2\\
&- \langle 1|y_s|0\rangle^2\left\{\frac{\langle 0,2|B|0,0\rangle^2}{\omega_1+2(\omega_2-\omega_3)}+\frac{\langle 0,-2|B|0,0\rangle^2}{\omega_1+2(\omega_2+\omega_3)}\right\} \qquad (35)
\end{aligned}
$$

where

$$
\begin{aligned}
\langle 0,0,0|h_4'|0,0,0\rangle &= -\frac{1}{\rho_{10}^2}-\frac{3}{\omega_2\rho_{10}^6}\left(\frac{1}{\omega_2}+\frac{1}{\omega_1}\right)+\frac{15}{8\rho_{10}^6}\left(\frac{1}{\omega_1}+\frac{1}{\omega_2}\right)^2\\
&+\frac{3}{8\rho_{10}^2}\left(1+\frac{\omega_2}{\omega_1}+\frac{1}{\omega_1\omega_2\rho_{10}^4}+\frac{3}{\omega_2^2\rho_{10}^4}\right)\\
&+\frac{3}{64\rho_{10}^2}\left(1-\frac{1}{\rho_{10}^4}\right)\left(\frac{13}{\omega_2^2}+\frac{12}{\omega_1\omega_2}+\frac{16}{\omega_1^2}\right) \qquad (36)
\end{aligned}
$$

$$
\langle 0,3|D|0,0\rangle = -\frac{5\sqrt{6}}{16\rho_{10}\omega_2^{3/2}}\left(1-\frac{1}{\rho_{10}^4}\right) \qquad (37)
$$

$$
\langle 0,0|B|0,0\rangle = -\frac{\sqrt{2}\omega_2}{2\rho_{10}} \qquad (38)
$$

$$
\langle 0,\pm 2|B|0,0\rangle = \pm\frac{1}{\rho_{10}^3}-\frac{\omega_2}{4\rho_{10}}-\frac{1}{4\omega_2\rho_{10}^3} \qquad (39)
$$

and the matrix elements involving y_s are ordinary oscillator matrix elements, that is $\langle 1|y_s|0\rangle = 1/\sqrt{\omega_1}$, etc. Note that, as mentioned above, $\epsilon_4 \to 0$ in both the $\beta \to 0$ and $\beta \to \infty$ limits.

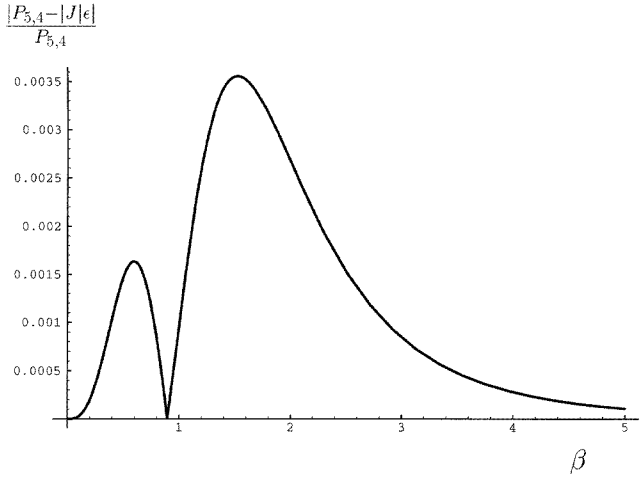


Figure 4. The relative difference between $|J|\epsilon$ and the $P_{5,4}$ Padé approximant for three electrons. The lowest antisymmetric state with $J = -3$ is studied.

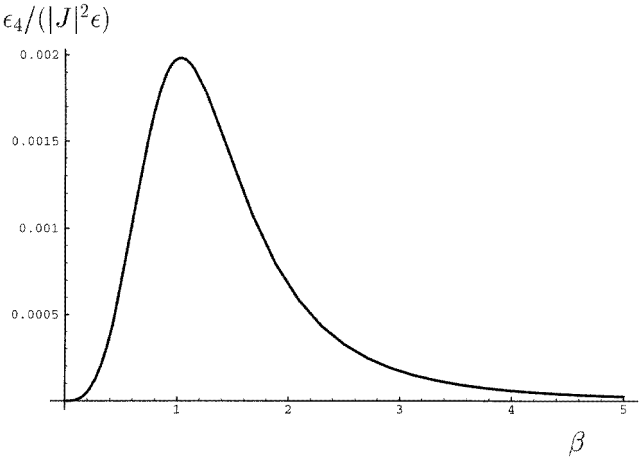


Figure 5. The relative weight of ϵ_4 in ϵ. The same state as in figure 4 is considered.

In figure 4 we compare the $1/|J|$-expansion and the two-point Padé approximant $P_{5,4}(\beta)$ of [14] for the lowest antisymmetric state with $J = -3$. As in the two-electron case, the relative error is of the same order as the error of the Padé estimate, i.e. a few parts in 10^3. The comparison with the numerical results of [10] for polarized states leads to similar conclusions. We show in figure 5 the relative weight of ϵ_4 in $\epsilon = \epsilon_0 + \epsilon_2/|J| + \epsilon_4/J^2$. For states with still higher values of $|J|$, the relative magnitude of the contribution of ϵ_4 decreases considerably.

5. Conclusions

We have computed the energy levels of two and three electrons in an ideal parabolic two-dimensional quantum dot in the presence of a magnetic field. States with high angular momentum and a low number of excitation (vibrational) quanta were considered. The energy was found as a series in the 'small' parameter $1/|J|$. The series exhibits good convergence properties even at $|J| = 3$. Notice that the expressions obtained for two electrons (equations (17–20)) can be used for any level, provided that $|J| \geqslant 3$ and n is not very high. In the three-electron case, we explicitly computed only the corrections ϵ_4 corresponding to the states with quantum numbers $(n_s, n, m) = (0, 0, 0)$ and arbitrary J (equations (35)–(39)). Calculations for other states are very similar and can be performed as simply as those sketched here.

At still lower values of the angular momentum the method is not applicable. However, we can obtain reliable analytic estimations by means of an alternative procedure: using the two-point Padé approximants, which are extremely simply constructed for the low-lying states [14]. $1/|J|$-expansions and Padé approximants were shown to give similar results at intermediate $|J|$-values, i.e. at $|J| = 3$.

Other problems are currently being treated along these lines. For example, the effects of anyonic statistics on the energy levels of two quasiparticles in a quantum dot may be straightforwardly studied by means of the $1/|J|$-method [16].

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